Synchronization of switching processes in coupled Lorenz systems

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Synchronization of two symmetrically coupled Lorenz systems, each of them considered a chaotic bistable system, is investigated numerically. A phenomenon of synchronization of the mean frequencies of switchings in coupled chaotic bistable systems is found. Bifurcations taking place in the system are analyzed. It is shown that there is the region on the ''coupling-detuning'' parameter plane where the mean frequencies of switchings coincide with a certain accuracy. [S1063-651X(97)12412-6]

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I. INTRODUCTION

As is already known, one of the mechanisms of self-organization in nonlinear oscillatory systems is synchronization. As a result of this phenomenon the interacting subsystems demonstrate the tendency to oscillate with equal or rationally related frequencies. In the case of weak coupling between subsystems the effect of frequency locking takes place and for strong coupling the suppression of one of the natural frequencies is observed [1]. Recent investigations have shown that similar effects take place when the interacting subsystems are irregular.

Synchronization of coupled chaotic oscillators has become the subject of much discussion in the past decade [2]. There are several approaches to the definition of synchronization of chaotic systems. In the simplest case, two identical chaotic systems are to be considered synchronized if their states coincide while the dynamics in time remains chaotic. This case was denoted ''full'' synchronization [3]. Another way to define chaotic synchronization was proposed by Pecora and Carroll [4]. They introduced into consideration the drive-response scheme and determined the conditions of synchronization by means of conditional Lyapunov exponents. In [5] the case of weakly coupled chaotic oscillators was considered. It was established that the interaction of nonidentical chaotic oscillators can lead to a perfect locking of their phases, whereas their amplitudes remain chaotic and uncorrelated.

In Refs. [6,7] the classical concept of synchronization was generalized to a certain class of chaotic systems for which the basic frequency is distinguishable in the power spectrum. In this case, chaotic synchronization is considered an interaction of natural time scales of subsystems that coincide at the moment of synchronization. In [8,9] it is shown that a similar interaction also takes place when these time scales are random values. The effect of the mean switching frequency locking was discovered in noisy bistable systems driven by a strong-amplitude periodic force [8]. It has been shown that there are regions on the ''noise intensity amplitude of periodic excitation'' parameter plane where the mean switching frequency remains constant and coincides with the frequency of the external periodic force.

The effect of stochastic synchronization of two coupled bistable systems driven by independent noise sources was discovered in [9]. It has been shown that the bifurcation of a two-dimensional stationary probability density takes place when the coupling is increased. Kramers's rates [10] of subsystems draw closer to one another when the coupling is increased and coincide at the bifurcation.

According to the results mentioned the effects of synchronization and synchronizationlike phenomena take place both in the case of an interaction of chaotic systems with clearly distinguishable time scales and in the case of an interaction of stochastic systems when the mean frequency of switchings plays the role of such time scales. It is reasonable to consider the interaction of chaotic dynamical systems, which may be considered chaotic bistable systems (Lorenz's system or Chua's circuit [11], for example). Switchings in such systems are caused by the natural chaotic dynamics and may be characterized by the mean frequency of switchings. The influence of the external periodic force on the process of switchings in Chua's circuit was considered in [12]. It was found that the effect of forced synchronization of switchings caused by the ''chaos-chaos'' intermittency takes place.

In this paper we investigate synchronization of switchings in two symmetrically coupled Lorenz systems, both of them being chaotic and bistable [13,14]. We use the well known ''model of two states'' [15] to describe the dynamics of the Lorenz system as a random process of switchings between two states. It has been shown that there is a region on the ''coupling-detuning'' parameter plane in which the processes of switchings in subsystems become coherent. We discuss the bifurcational mechanism of this phenomenon.

In Sec. II we present the model and results of numerical experiments. In Sec. III we discuss bifurcations of saddle cycles and equilibrium states that take place in the system and their relation with the synchronization of the processes of switching in subsystems. The construction of the synchronization region for mean frequencies of switchings is described in Sec. IV. Conclusions are given in Sec. V.

II. MODEL AND NUMERICAL SIMULATION

The dynamical system under study is

$$\dot{x}_1=\sigma(y_1-x_1)+\gamma(x_2-x_1),$$
$$\dot{y}_1=r_1x_1-x_1z_1-y_1,$$
$$\dot{z}_1=x_1y_1-z_1b, \tag{1}$$
$$\dot{x}_2=\sigma(y_2-x_2)+\gamma(x_1-x_2),$$
$$\dot{y}_2=r_2x_2-x_2z_2-y_2,$$
$$\dot{z}_2=x_2y_2-z_2b.$$

The parameters of system (1) are $\sigma=10$, $r_1=28.8$, $r_2=28$, and $b=8/3$ and the Lorenz attractor exists in each subsystem. When

$$x_1'=\begin{cases}+1, & x_1>0\\ -1, & x_1<0,\end{cases}\quad x_2'=\begin{cases}+1, & x_2>0\\ -1, & x_2<0\end{cases}$$

model (1) might be considered a system of two symmetrically coupled bistable subsystems [14]. Switchings in subsystems are caused by the natural chaotic dynamics of Lorenz systems. The process of switchings can be characterized by the residence time probability density $p(\tau)$ [16] in one of the states, marked $+1$ and -1. Analogously to a stochastic bistable system, it is possible to introduce the mean frequency of transitions from one state to another. The mean period of switchings being the first moment of $p(\tau)$ can be determined as follows

$$\langle T\rangle=\int_0^\infty \tau p(\tau)d\tau.$$

Hence the mean frequency of switchings is $\langle f\rangle=2\pi/\langle T\rangle$. Here and throughout this paper, the mean frequency of switchings is considered a natural statistical time scale of subsystems. The mean frequencies of switchings $\langle f_1\rangle,\langle f_2\rangle$ versus coupling are shown in Fig. 1. In the case of $\gamma=0$, the frequencies of switchings are different because of the detuning between r_1 and r_2. The increase of parameter r leads to the growth of the mean frequency of switchings. Frequencies $\langle f_1\rangle$ and $\langle f_2\rangle$ decrease when coupling increases. They reach a minimum value at $\gamma\approx 2.1$ (this peculiarity in the behavior of the mean frequencies will be disscused below in Sec. III). A further increase of the coupling leads to the frequencies $\langle f_1\rangle$ and $\langle f_2\rangle$ approaching each other and coinciding at $\gamma=6.0$. The common mean frequency of switchings does not coincide with the initial value of either $\langle f_1\rangle$ or $\langle f_2\rangle$.

Thus, when the coupling grows the mean frequencies of switchings draw closer to one another and coincide for some critical value of coupling. Such behavior is typical for frequencies of coupled regular oscillators. Analogously to the classical theory of oscillations, this effect might be called *synchronization of the processes of switchings in two symmetrically coupled chaotic bistable systems.*

The coherence degree of the processes of switchings in subsystems can be estimated by means of coherent function

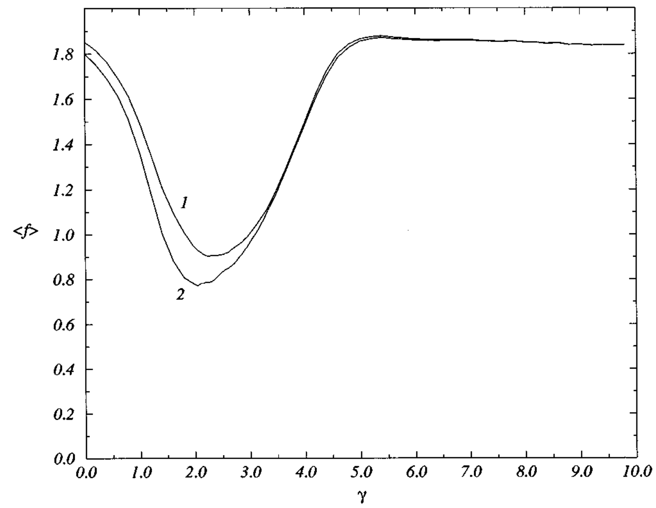

FIG. 1. Mean frequencies of switchings 1 ($\langle f_1\rangle$) and 2 ($\langle f_2\rangle$) vs coupling parameter γ.

$$\Gamma(\omega)=\frac{|S_{x_1'x_2'}(\omega)|}{\sqrt{S_{x_1'}(\omega)S_{x_2'}(\omega)}},$$

where $S_{x_1'}(\omega)$ and $S_{x_2'}(\omega)$ are the power spectra of the processes $x_1'(t)$ and $x_2'(t)$, respectively, and $S_{x_1',x_2'}(\omega)$ is the mutual spectrum of the processes $x_1'(t)$ and $x_2'(t)$. It is well known that the coherence function varies from zero to unity. The approach of Γ to unity within some interval of frequencies testifies to the growth of coherence in this interval. The results of calculations are presented in Fig. 2. In the case of zero coupling $\Gamma\approx 0$ (curve 1 in Fig. 2). When coupling is increased the coherence function in the low-frequency domain grows. This demonstrates the growth of the coherence degree between the processes of switchings in subsystems. The process of synchronization may be illustrated also by means of Lyapunov exponents and mutual phase projections of phase trajectories, which are presented in Figs. 3 and 4, respectively. It is clearly seen in Fig. 3(a) that the second

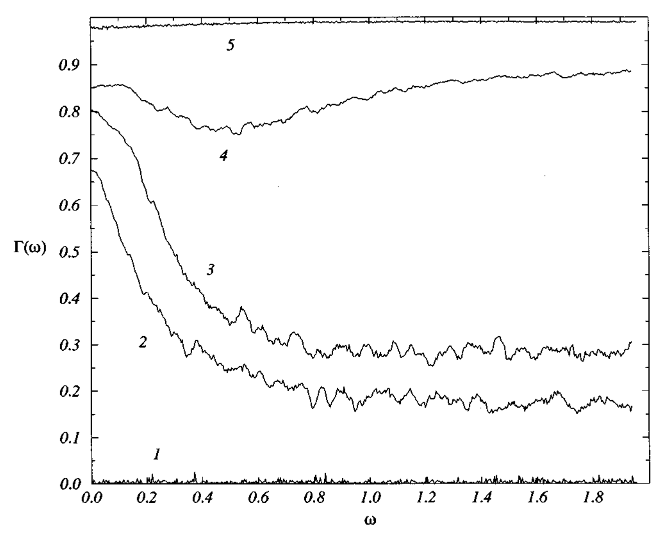

FIG. 2. Coherence function $\Gamma(\omega)$ for different values of coupling: 1, $\gamma=0.0$; 2, $\gamma=1.0$; 3, $\gamma=2.1$; 4, $\gamma=4.05$; and 5, $\gamma=6.0$.

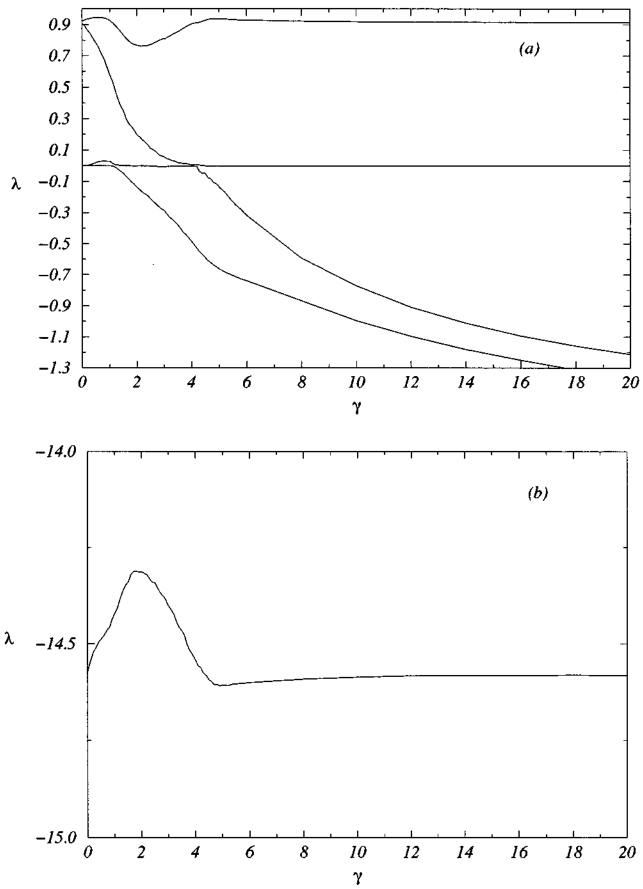

FIG. 3. Lyapunov exponents λ vs coupling parameter: (a) the four largest Lyapunov exponents (the positive Lyapunov exponent is saturated for strong coupling) and (b) the saturation of one of the negative Lyapunov exponent. It should be noted that in the case of an interaction of identical subsystems the Lyapunov exponents demonstrate similar behavior and two of them are saturated with the growth of the coupling parameter.

Lyapunov exponent changes its sign at $\gamma \approx 4$. At this moment the mean frequencies of switchings are nearly equal to each other (Fig. 1), although the excursions from the line $x_1 \approx x_2$ are frequent, as seen in Fig. 4(b). Further growth of coupling causes the coincidence of the frequencies of switchings and the saturation of two Lyapunov exponents, as seen in Figs. 3(a) and 3(b). For strong coupling the excursions from the line $x_1 \approx x_2$ become impossible and phase trajectories do not leave a certain small-δ neighborhood of symmetrical subspace [Fig. 4(d)]. Thus, when the coupling is strong enough the Lorenz-type attractor exists in a certain small-δ neighborhood of symmetrical subspace, which might be considered a mathematical image of synchronous oscillations in the system. The above results of numerical simulation illustrate the qualitative changes in system (1); however, they do not touch upon the bifurcational mechanism of the phenomenon of mutual synchronization of switchings, which is considered in the following section.

III. BIFURCATIONAL ANALYSIS OF THE SYSTEM

It is known that the Lorenz attractor includes a denumerable set of saddle cycles that surround a pair of symmetrically located equilibrium states [17]. The existence of these saddle cycles causes the residence time probability density to have a clearly distinguishable structure [Fig. 5(a)]. The results of investigations show that the first peak in the residence time probability density corresponds to the saddle

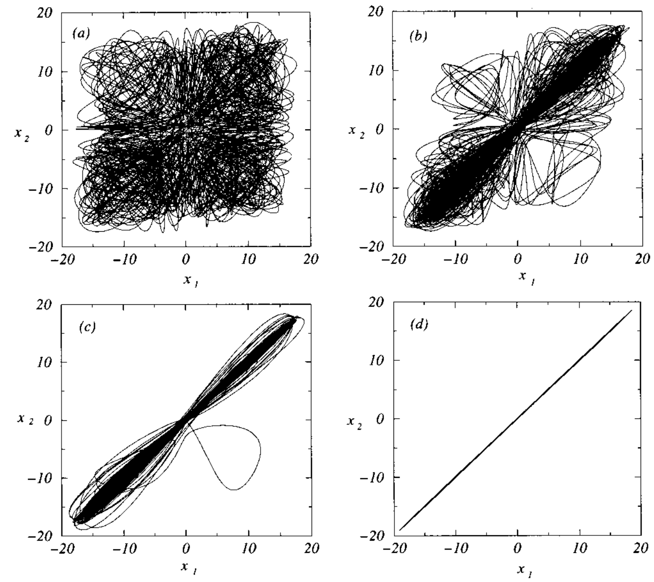

FIG. 4. (x_1, x_2) phase projections of phase trajectories for different values of coupling: (a) $\gamma = 1.0$, the case of weak coupling; (b) $\gamma = 4.05$, the beginning of synchronization of switchings; (c) $\gamma = 6.0$; and (d) $\gamma = 30$, synchronous oscillations of subsystems. The phase trajectory (d) does not leave the δ neighborhood of symmetrical subspace. (δ is smaller than the typical scales of the attractor and in our case $\delta \sim 10^{-1}$.)

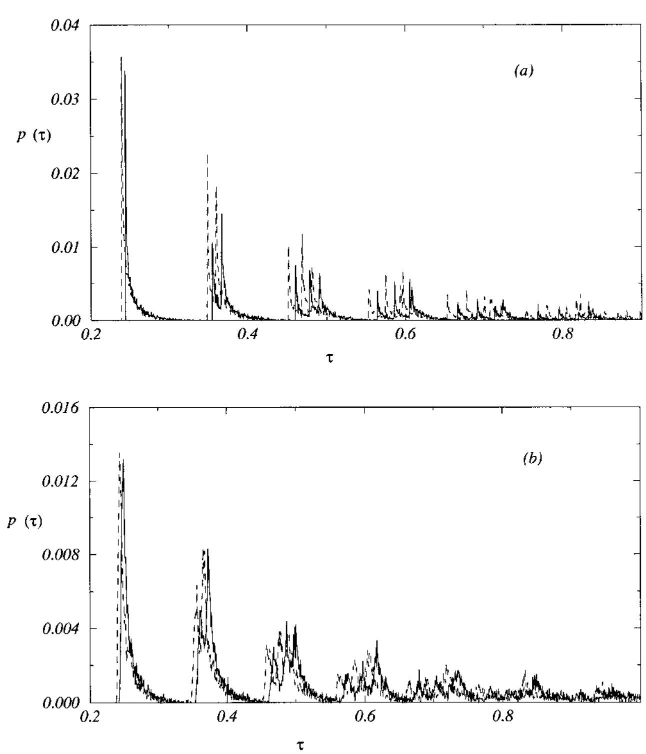

FIG. 5. Residence time probability densities $p(\tau)$ for different values of the coupling parameter: (a) $\gamma = 0.0$ and (b) $\gamma = 0.2$.

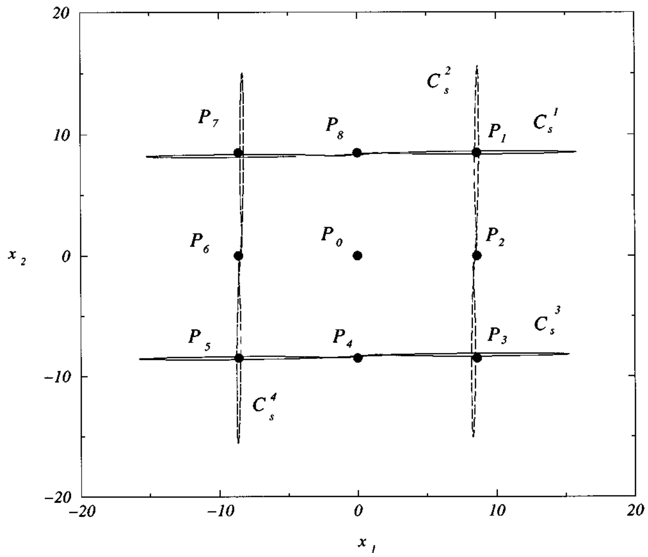

FIG. 6. One-turn cycles belonging to the families C_s^1, C_s^3 (solid line) and C_s^2, C_s^4 (dashed line). Black full circles show the position of equilibrium states P_0–P_8.

cycle, which makes one rotation around each of the equilibrium states. It is called a one-turn cycle. All the consequent reasonings about the evolution of some family of saddle cycles will be derived from the information about the evolu-

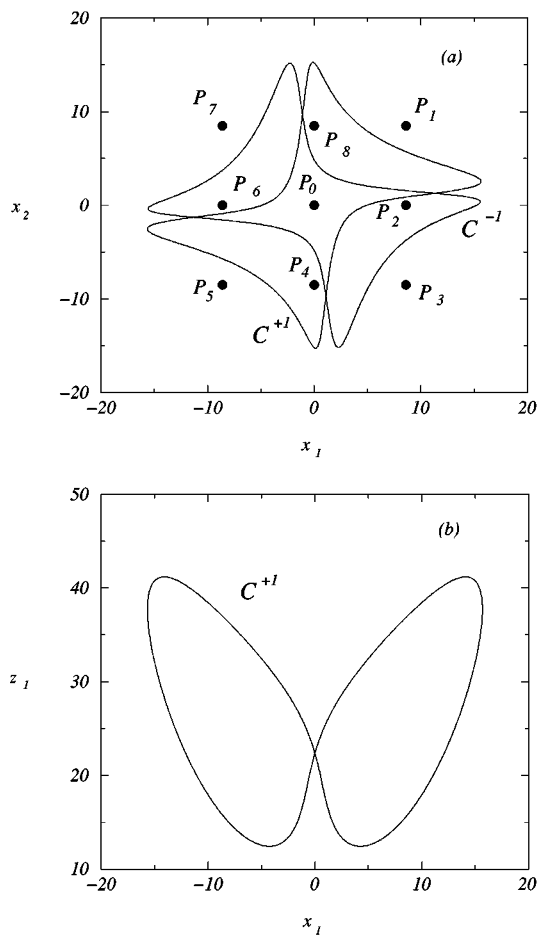

FIG. 7. One-turn cycles C^{+1} and C^{-1}: (a) (x_1,x_2) projection and (b) (x_1,z_1) projection on the plane. Filled circles show the positions of the equilibrium states P_0–P_8.

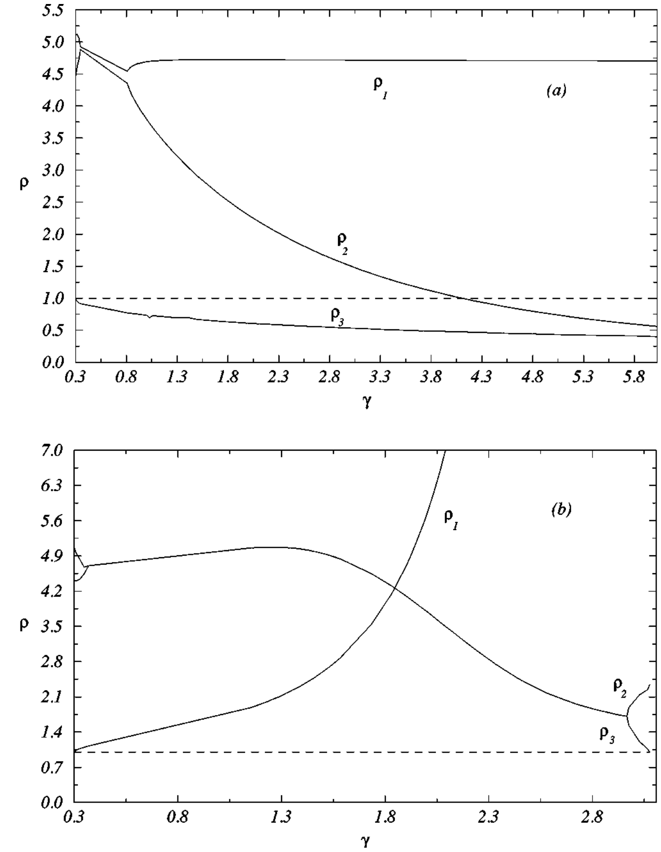

FIG. 8. Multipliers ρ_1, ρ_2, ρ_3 vs coupling parameter for (a) the saddle cycle C^{+1} and (b) the saddle cycle C^{-1}. Every saddle cycle in R^6 has six multipliers, but we consider only the three largest because one of the three others is equal to unity and the remaining ones are of order 10^{-4}.

tion of the one-turn cycle, which belongs to this family, because all the other saddle cycles of this family undergo the same bifurcations as the appointed one-turn cycle. For the case of zero coupling the residence time probability densities are presented in Fig. 5(a) (the residence time probability density for the first subsystem is marked by the solid line and the one for the second subsystem by the dashed line). The displacement of the peaks in the residence time probability density can be explained by the detuning of the parameters of the subsystems.

Let us consider the qualitative changes of the structure of phase space R^6 when coupling is introduced. In the case of weak coupling ($\gamma < 0.2$), there exist four families of saddle cycles $C_s^1, C_s^2, C_s^3, C_s^4$, which define the structure of phase space R^6. The one-turn cycles from these families are presented in Fig. 6. We should note that the families mentioned above coincide in pairs with the families that exist in the subsystems when the coupling vanishes (families C_s^1, C_s^3 coincide with the family in the first subsystem and C_s^2, C_s^4 with the family in the second subsystem, respectively). Hence, in the case of weak coupling the mean period of switchings in each subsystem is defined by the family of saddle cycles existing in the subsystem when the coupling is equal to zero. The mean frequencies of switchings are different, while the probabilities of visiting different regions of phase space are almost the same.

When the coupling reaches the value $\gamma = 0.273\,51$ the tangent bifurcation takes place in the system. Due to this bifurcation the pair of saddle cycles C^{+1},C^{-1} forms (Fig. 7). The three largest multipliers of cycles C^{+1} and C^{-1} versus the parameter γ are presented in Figs. 8(a) and 8(b), respectively. It is clearly seen that the saddle cycles C^{+1} and C^{-1} evolve in different ways. The cycle C^{+1} contracts to the symmetrical subspace ($x_1=x_2$, $y_1=y_2$, and $z_1=z_2$) and becomes stable in four directions (the dimension of the stable manifold increases from 3 to 4) with the growth of the coupling, whereas the saddle cycle C^{-1} becomes unstable and disappears due to tangency bifurcation (one of the multipliers becomes $+1$). Numerical simulation shows that a denumerable set of pairs of saddle cycles is formed in the system when the coupling increases. They form the two families of saddle cycles C_n^{+1} and C_n^{-1}. The family C_n^{+1} is located in the neighborhood of symmetrical subspace and becomes more attractive when the parameter of coupling increases (the dimensions of the stable manifolds of all saddle cycles belonging to this family increases), while the family C_n^{-1} becomes unstable and disappears with the growth of coupling. It is possible to regard that this family disappears as $\gamma=3.04$ because the one-turn cycle C^{-1} finally disappears.

The above results can be illustrated by means of the residence time probability density (Figs. 5 and 9). For weak coupling $\gamma=0.2$ [Fig. 5(b)] subsystems respond to each other as to small perturbations and the changes in the probability densities are also very small. The growth of coupling leads to the destruction of the structures in the residence time probability densities. When the coupling reaches the value of $\gamma=4.05$ [Fig. 9(a)] the structures in the residence time probability densities become similar to each other and the multiplier of the saddle cycle C^{+1} becomes equal to unity as clearly seen from Fig. 8(a). Moreover, the second Lyapunov exponent changes its sign at this moment, thus allowing us to consider it as the beginning of synchronization of the processes of switchings in the subsystems.

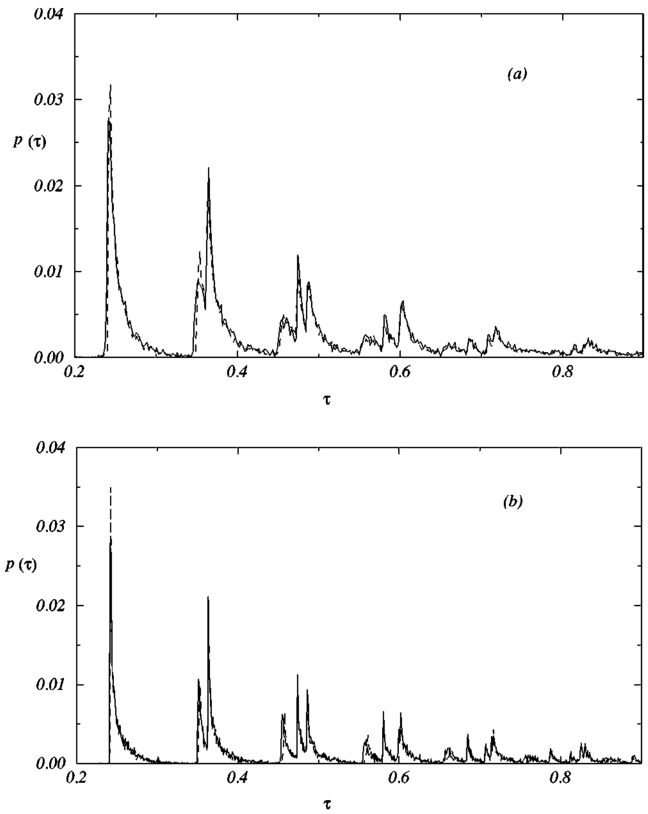

FIG. 9. Residence time probability densities $p(\tau)$ for different values of the coupling parameter: (a) $\gamma = 4.05$ and (b) $\gamma = 6.0$.

Along with the bifurcations of saddle cycles considered above, Hopf's bifurcations in the neighborhoods of the equilibrium states take place. System (1) has nine equilibrium states (Fig. 6), which have the following coordinates for zero coupling:

$$P_0:(0,0,0,0,0,0),\quad P_1:(a_1,a_1,a_3,a_2,a_2,a_4),\quad P_2:(a_1,a_1,a_3,0,0,0),\quad P_3:(a_1,a_1,a_3,-a_2,-a_2,a_4),$$

$$P_4:(0,0,0,-a_2,-a_2,a_4),\quad P_5:(-a_1,-a_1,a_3,-a_2,-a_2,a_4),\quad P_6:(-a_1,-a_1,a_3,0,0,0),$$

$$P_7:(-a_1,-a_1,a_3,a_2,a_2,a_4),\quad P_8:(0,0,0,a_2,a_2,a_4),$$

$$a_1=\sqrt{b(r_1-1)},\;\; a_2=\sqrt{b(r_2-1)},\;\; a_3=r_1-1,\;\; a_4=r_2-1$$

.

Local properties of the flow in the neighborhoods of the equilibrium states are investigated using well-known computer program LOCBIF [18].

Investigations have shown that equilibrium states P_0–P_8 undergo in pairs Hopf's bifurcations and saddle cycles are formed when coupling is introduced. The equilibrium states P_1,P_5 undergo bifurcation at $\gamma=0.515$; P_3,P_7 at $\gamma=0.7944$; P_4,P_8 at $\gamma=1.191$; and P_2,P_6 at $\gamma=1.473$. These bifurcations do not lead to the global reconstruction of the structure of phase space; however, the saddle cycles that arise via bifurcations influence the absolute value of the mean period of switchings. The saddle cycles C_1^0 and C_2^0 are formed in the neighborhoods of the equilibrium states P_1 and P_5, respectively, and become stable in the four directions [the three largest multipliers from the coupling of cycle C_2^0 are presented in Fig. 10(b)]. As a result, cycles C_1^0 and C_2^0 turn out to be the most attractive cycles among the saddle cycles that are located in the neighborhood of symmetrical subspace (the dimensions of the stable manifolds of saddle cycles that belong to the family C_n^{+1} are equal to 3 for weak coupling, while the dimensions of the stable manifolds of the

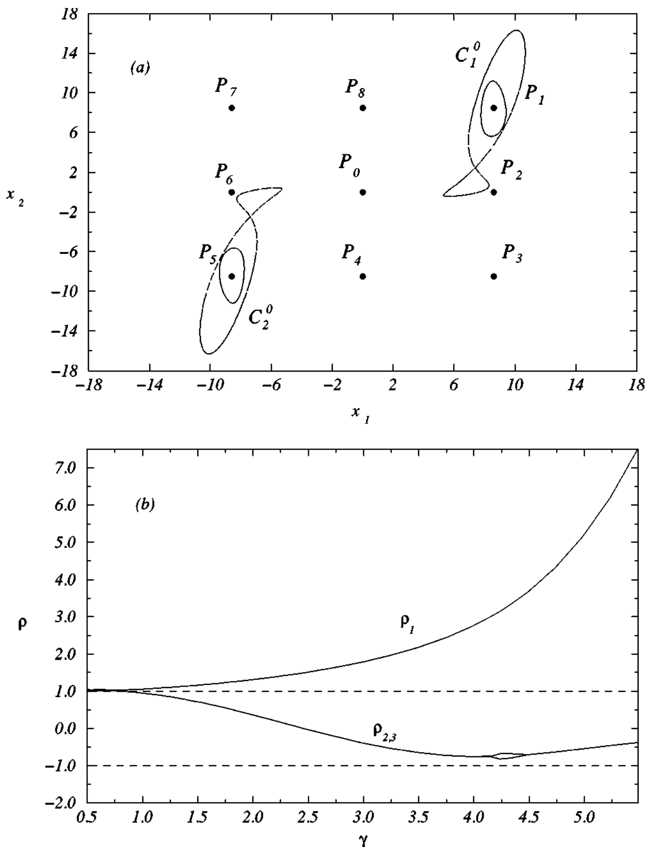

FIG. 10. (a) (x_1,x_2) projections of the saddle cycles C_1^0 and C_2^0 (the solid line is for $\gamma = 0.9$ and the dashed line is for $\gamma = 4.8$) and (b) dependence of the multipliers of the saddle cycle C_2^0 vs the coupling parameter.

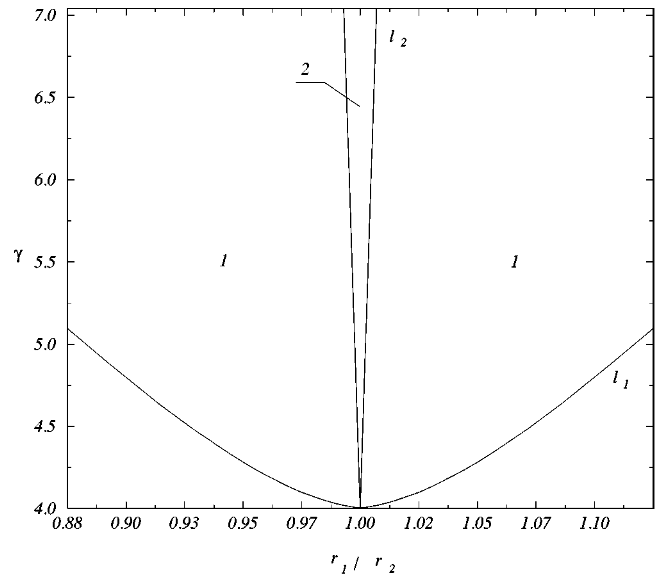

FIG. 11. Regions of synchronization: 1 is the region of synchronization of the processes of switchings and 2 is the region of synchronous oscillations.

cycles C_1^0 and C_2^0 are equal to 4). Because of the twisting of phase trajectories around these saddle cycles (C_1^0 and C_2^0) the residence times in both states $+1$ and -1 increases and, as a result, the absolute value of the mean frequencies of switchings decreases (Fig. 1). The increase of coupling ($0.27 < \gamma < 3.04$) leads to a number of bifurcations resulting in the formation of saddle cycles. The formation of saddle cycles in R^6 with the growth of coupling leads to the more effective mixing and causes the destruction of structures in the residence time probability densities.

It was mentioned above that the family of saddle cycles C_n^{-1} disappears at $\gamma = 3.04$ and the saddle cycles that belong to the families $C_s^1, C_s^2, C_s^3, C_s^4$ are unstable with respect to asymmetric perturbations. Saddle cycles C_1^0 and C_2^0 that are also unstable to asymmetric perturbations increase in size and rotate around the symmetrical subspace [Fig. 10(a)]. A further increase of coupling leads to the growth of dimensions of stable manifolds of the saddle cycles belonging to the family C_n^{+1} (the dimension of the stable manifold of the one-turn cycle C^{+1} becomes equal to 4 as $\gamma = 4.05$). As a consequence, the neighborhood of the symmetrical subspace becomes the most frequently visited region in the phase space. For these reasons, the growth and approach of the mean frequenies of switchings can be observed.

Thus bifurcational analysis of system (1) shows that the bifurcations of saddle cycles play the key role in the reconstruction of the structure of the phase space. As a result of these bifurcations a family of saddle cycles C_n^{+1} is formed in the neighborhood of symmetrical subspace. This family becomes the most attractive family of saddle cycles in the phase space. Therefore, the probability of the hits of trajectories in the neighborhood of symmetrical subspace becomes largest and the probability of the coherent switchings increases.

IV. REGION OF SYNCHRONIZATION OF SWITCHINGS

It is well known from the classical theory of oscillations that the fundamental characteristic of synchronization is a region of coherent behavior of subsystems on the coupling-detuning plane. Therefore, we try to construct a similar region of synchronization for two coupled bistable dynamical systems. The parameter r is chosen as the control parameter, which allows us to change the mean frequency of switchings in the subsystem. The detuning of subsystems in this case is $p = r_1/r_2$, where r_1, r_2 are the parameters of the first and second subsystems, respectively. The time scales that characterize subsystems are the random values (mean frequencies of switchings $\langle f_1 \rangle, \langle f_2 \rangle$) and it is difficult to speak about the construction of the synchronization region on the coupling-detuning plane in the classical sense. It is reasonable to construct the region where the frequencies $\langle f_1 \rangle$ and $\langle f_2 \rangle$ coincide with some accuracy. As mentioned above, the mean frequencies of switching $\langle f_1 \rangle, \langle f_2 \rangle$ become nearly equal to each other (they differ by 0.8%) at the moment when one of the multipliers of the one-turn cycle C^{+1} becomes equal to unity and one of the Lyapunov exponents changes its sign. Taking into account these results, it is natural to consider the bifurcation line l_1 (Fig. 11) for the saddle cycle C^{+1} on the coupling-detuning parameter plane as the boundary of the region of switching synchronization. Moreover, it is possible to build the region on the same parameter plane, inside of which the states of oscillators are very close to each other and oscillations of nonidentical subsystems can be considered synchronized. The construction of such region requires some preliminary explanation. As is already known, in the case of an interaction of identical oscillators, the integral manifold exists in symmetrical subspace. The growth of coupling causes the asymptotic stability of this manifold, which leads to the full synchronization of subsystems. The introduction of parameter detunings is equivalent to the introduction of some perturbation in the system of identical oscillators and thus the system of coupled nonidentical oscillators is equivalent to the perturbed system of coupled identical oscillators. As known from the theory of integral manifolds [19], for perturbed system the local integral manifold exists in a certain small neighborhood of the integral manifold of the nonperturbed system. This local integral manifold becomes stable when the coupling increased (line l_2 in Fig. 11) and the phase trajectory does not leave the small neighborhood of symmetrical subspace, causing, in particular, the saturation of Lyapunov exponents.

V. CONCLUSION

In this work we have investigated two symmetrically coupled chaotic bistable systems. Switchings in such systems are caused by the natural chaotic dynamics. The phenomenon of mutual synchronization of mean frequencies of switchings in subsystems is found. The growth of the coherence degree between switchings in subsystems is accompanied by the qualitative reconstruction of the structure of phase space. This reconstruction is due to the bifurcations of equilibrium states and saddle cycles that take place in the system. As a result of tangent bifurcations, the family of saddle cycles is formed in some neighborhood of symmetrical subspace. When coupling increases this family becomes more attractive, which causes the growth of the probability visiting of the neighborhood of symmetrical subspace and leads to synchronization of the processes of switchings. Hopf's bifurcations in the neighborhoods of equilibrium states, which also take place in the system, do not lead to the global rebuilding of the attractor; however, they influence the mean period of switchings $\langle T \rangle$.

We have been found that there is a region on the detuning-coupling parameter plane in which the mean frequencies of switchings coincide within the limit of some accuracy and switchings in subsystems are coherent. This region includes the domain inside of which the oscillations of nonidentical subsystems might be considered synchronized. Hence we conclude that synchronization of coupled chaotic bistable systems begins from the synchronization of the processes of switchings in subsystems.

ACKNOWLEDGMENTS

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[1] I. I. Blekhman, *Synchronization of Dynamical Systems* (Nauka, Moscow, 1971).

[2] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); A. S. Pikovsky, Z. Phys. B **55**, 149 (1984); P. S. Landa and M. G. Rosenblum, Appl. Mech. Rev. **46**, 414 (1993); L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).

[3] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **29**, 795 (1986).

[4] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[5] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[6] V. S. Anishchenko, T. E. Vadivasova, D. E. Postnov, and M. A. Safonova, Int. J. Bifurcation Chaos **2**, 644 (1992).

[7] V. S. Anishchenko, *Dynamical Chaos—Models and Experiments* (World Scientific, Singapore, 1995).

[8] B. V. Shulgin, A. B. Neiman, and V. S. Anishchenko, Phys. Rev. Lett. **75**, 4157 (1995).

[9] A. B. Neiman, Phys. Rev. E **49**, 3484 (1994).

[10] H. A. Kramers, Physica **7**, 28 (1940).

[11] V. S. Anishchenko, A. B. Neiman, and M. A. Safonova, J. Stat. Phys. **70**, 183 (1993).

[12] I. A. Khovanov and V. S. Anishchenko, Tech. Phys. Lett. **2**, 854 (1996).

[13] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[14] C. Nicolis and G. Nicolis, Phys. Rev. A **34**, 2384 (1986).

[15] V. S. Anishchenko, M. A. Safonova, and L. O. Chua, J. Circuits Syst. Comp. **3**, 553 (1993).

[16] T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).

[17] V. S. Afraimovich, V. V. Bykov, and L. P. Shil'nikov, Trans. Moscow Soc. **44**, 153 (1982).

[18] A. I. Khibnik, Yu. A. Kuznetsov, V. V. Levitin, and E. V. Nikolaev, Physica D **62**, 360 (1992).

[19] U. A. Mitropolsky and O. B. Lykova, *The Integral Manifolds in Nonlinear Mechanics* (Nauka, Moscow, 1973).